LYAPUNOV-STABLE EQUILIBRIUM DISTRIBUTION OF CHARGED PARTICLES IN AN IONIZED-GAS CLOUD

S. V. Temko

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The solution of several problems requires establishing the equilibrium spatial distribution of the charged particles in an ionized-gas cloud having fixed shape and given dimensions.

In the present study we find the Lyapunov-stable equilibrium spatial distribution of the charged particles in an ionized-gas cloud enclosed in a region of space with fixed form and given dimensions. An expression is obtained for the limiting energy stored by the ionized gas for given degree of ionization of neutral gas, and expressions are found for the pressure and chemical potential of the ionized gas in the cloud. We use the general method for the solution of a particular type of extremal problem presented in [1, 2], which reduces the optimization problem to the solution of the corresponding inverse problem of potential theory.

1. Let the ionized-gas cloud have the form F, corresponding to an ellipsoid of revolution. We take one of the ellipsoid axes as the axis of revolution. We consider only the case of Lyapunov-stable equilibrium accumulation of charged particles confined in the elliptical region F, and we neglect the proper rotation of the cloud as a whole about one of the axes of the ellipsoid F. Since an elliptical region of space belongs to the class of convex regions, its boundary satisfied the well-known Poincaré condition [3], i.e., we can consider each point of the boundary of the region F as the apex of a cone which is located entirely inside the given three-dimensional region F.

Now let us find the Lyapunov-stable equilibrium charged-particle distribution in the ionized-gas cloud having elliptical form and given dimensions. To do this we introduce the charged particle distribution in the ionized-cloud, denoting this distribution by μ. We seek the distribution μ in the form

$$d\mu = \chi(Q)\, dv_Q. \tag{1.1}$$

Here dv_Q is a three-dimensional volume element; $\chi(Q)$ is the charged-particle distribution density in the ionized gas; Q is a point belonging to the elliptic spatial region F filled with the ionized gas.

Hereafter we use $Q \in F$ to denote that the point Q belongs to the region F.

We say that the charge distribution μ is concentrated in the fixed spatial region F, if there are no ionized gas particles outside the region F. This circumstance is denoted by the symbol $\mu \prec F$

Let $\varphi(r)$ be the potential function corresponding to the interaction energy of a pair of unit charges located at the distance r from one another. It was shown in [1] that the equilibrium theorem is valid for a broad class of potential functions $\varphi(r)$.

A detailed analysis of the potential functions for which the equilibrium theorem is valid was made in [1, 2].

For the sake of simplicity, in the present study we take as the potential function that corresponding to the "smoothed" Coulomb interaction

$$\varphi(r) = r^{-1}(1 - e^{-\lambda r}), \tag{1.2}$$

which accounts approximately for the short-range forces [4–6]. Our analysis is presented basically for the case in which the short-range interaction parameter λ^{-1} is not small and is comparable with the average distance between the ionized gas particles in the cloud.

Since $\varphi(r)$ taken in the form (1.2) is a convex function, the equilibrium theorem from [1] is valid for it.

We use $|P - Q|$ to denote the distance between the points P and Q. Let $P \in F$, $Q \in F$ and $\mu \prec F$; then the functional

$$I(\mu)=\int_F\int_F \varphi(|P-Q|)\chi(P)\chi(Q)\,dv_P dv_Q \tag{1.3}$$

(here the integration is performed over the entire elliptic region F filled with the ionized gas) defines the potential energy of the entire ionized gas cloud having fixed form F and given dimensions for the charged particle distribution μ represented in the form (1.1) and for the selected potential function $\varphi(r)$.

The system of interacting particles tends [7] to take a configuration which minimizes its potential energy. The state of dynamic equilibrium of the particle ensemble for which the potential energy $I(\mu)$ of the entire ensemble takes the minimum value corresponds to Lyapunov-stable equilibrium of the given ensemble [8].

Let W(F) be the minimum value of the potential energy of the ionized-gas cloud of elliptic form and given dimensions. In accordance with [1]

$$W(F)=\inf_{\mu<F} I(\mu), \tag{1.4}$$

where the greatest lower bound is taken over all possible charged-particle distributions μ in the ionized-gas cloud F.

Following [1], we introduce the φ-capacitance of the spatial region F.

By the φ-capacitance of the region F we mean [1] the number $C(F;\varphi) > 0$, which for a finite value of the potential energy W(F) is found from the equation

$$W(F)= K\varphi[c(F;\varphi)], \tag{1.5}$$

where K is a coefficient of proportionality. In our case the φ-capacitance $C(F;\varphi)$ corresponds to the electrical capacitance of the ionized-gas cloud of elliptical form F. If $W(F) = +\infty$, then the φ-capacitance of the region F is assumed to be zero. This corresponds to the case in which, for example, the entire ionized gas is concentrated at a single point.

The elliptical region F with finite dimensions has positive φ-capacitance; therefore, as a result of [1] for the given region F there exists a unique equilibrium spatial charged-particle distribution μ^*, $d\mu^* = \chi^*(Q)\,dv_Q$ in the ionized gas cloud, which realizes a minimum of the potential energy $I(\mu)$, represented by the functional (1.3).

We term this charged particle distribution in the ionized-gas cloud the Lyapunov-stable equilibrium spatial distribution of the ionized gas in a cloud with fixed form F and given dimensions.

We note that the ionized gas in the Lyapunov-stable equilibrium state in a finite space region F is in reality concentrated in the region F_{μ^*}, termed the support of the distribution μ^*.

In general, the support of the distribution μ is that region $F\mu$ of the given region F for which each point P has the following property: no matter what the vicinity O(P) of the point P the requirement that $\mu[O(P)] \neq 0$ is met. We note that the region F_μ^* is found by minimization of the functional $I(\mu)$.

2. To find the Lyapunov-stable equilibrium charged-particle distribution μ^* in an ionized-gas cloud, which minimizes the function $I(\mu)$, we introduce the φ-potential u(P), generated by the distribution μ. In accordance with [1] the φ-potential u(P), generated by the distribution represented in the form (1.1), is defined by an integral of the following form:

$$u(P)=\int_F \varphi(|P-Q|)\chi(Q)\,dv_Q \tag{2.1}$$

If the φ-potential u(P) is generated by a Lyapunov-stable distribution μ^*, it is termed the equilibrium φ-potential and is denoted by $u^*(P)$ (see [1]).

Since the potential function $\varphi(r)$, represented in the form (1.2), satisfies the equilibrium theorem from [1], for the elliptic region F of positive φ-capacitance the equilibrium charged-particle distribution μ^* generates the equilibrium φ-potential $u^*(P)$ having the following properties:

1) the potential $u^*(P) = W(F)K^{-1/2}$ for all points $P \in F$,

2) the potential $u^*(P) \leq W(F)K^{-1/2}$ for all P located outside the region F ($P \overline{\in} F$).

Thus, for an elliptic region filled by an ionized gas cloud whose boundary satisfies the Poincaré condition, at all points $P \in F$ and $\mu^* \prec F$ we have $u^*(P) = W(F)K^{-1/2}$. Hence follows

$$W(F)K^{-1/2} = \int_F \varphi(|P-Q|)\chi^*(Q)dv_Q. \tag{2.2}$$

We introduce the auxiliary function V(P), defining it as follows:

$$V(P) = \begin{cases} W(F) & (P \in F) \\ 0 & (P \overline{\in} F) \end{cases}.$$

Then we obtain the following integral equation of the first kind for finding the charged-particle distribution density $\chi(Q)$ in an ionized gas in the present case:

$$V(P)K^{-1/2} = \int_F \varphi(|P-Q|)\chi(Q)dv_Q \tag{2.3}$$

It can be shown that the distribution μ satisfying an equation of the (2.3) type minimizes functional (1.3).

3. To solve the integral equation (2.3) we use the general method for finding the Lyapunov-stable equilibrium distribution μ^* for the selected potential function $\varphi(r)$ and the fixed region F with given dimensions, based on the results of [1, 2]. We apply the Fourier transform to (2.3) and obtain the following operator equation:

$$V^0(P)K^{-1/2} = (2\pi)^{3/2}\varphi^0(P)\chi^0(P). \tag{3.1}$$

Here and hereafter the superscript 0 denotes Fourier transforms of the functions. For the given potential function $\varphi(r)$ equality (3.1) is completely valid.

We obtain for the Fourier transform of the function V(P)

$$V^0(P) = W(F)\psi^0(P;F). \tag{3.2}$$

In this relation $\psi^\circ(P;F)$ is the Fourier transform of the characteristic function of the region F,

$$\psi^0(P;F) = \frac{1}{(2\pi)^{3/2}}\int_F e^{i(PM)}dv_M. \tag{3.3}$$

Here (PM) is the scalar product of the radii-vectors OP and OM of the points $P = (x_1, x_2, x_3)$ and $M(y_1, y_2, y_3)$.

Since $\varphi(|M|) = \varphi(r)$ $(r^2 = y_1^2 + y_2^2 + y_3^2)$, in view of [9] we have

$$\varphi^0(P) = \varphi^0(\rho) \quad (\rho^2 = x_1^2 + x_2^2 + x_3^2).$$

Here $\varphi^\circ(\rho)$ is the Hankel transform of the function $\varphi(z)$:

$$\varphi^0(\rho) = \frac{1}{\sqrt{\rho}}\int_0^\infty r^{3/2}\varphi(r)J_{1/2}(r\rho)dr, \tag{3.4}$$

where $J_\nu(t)$ is a Bessel Function of the first kind of order ν.

Now the operator equation for $\chi^\circ(P)$ is written in the form

$$W(F)\psi^0(P;F)K^{-1/2} = (2\pi)^{3/2}\varphi^0(\rho)\chi^0(P), \tag{3.5}$$

and the problem of finding $\chi^*(Q)$ reduces to finding the Fourier original for the function $\psi^\circ(P;F)/\varphi^\circ(\rho)$. We note that in the case of a region of arbitrary form the Fourier original for $\chi^\circ(P)$ can be found by numerical methods [10].

4. With the aid of the operator equation (3.5) we find the explicit form for $\chi^*(Q)$ in the case of an elliptic region F

in which

$$a_1^2x_1^2 + a_2^2x_2^2 + a_3^2x_3^2 \leqslant 1 \quad (a_1,\ a_2,\ a_3 > 0). \tag{4.1}$$

(For the ellipsoid of revolution $a_1 = a_2 = a$, $a_3 = c$.)

Let us first find the explicit form for $\psi^\circ(P; F)$. After conversion to the new variables $t_k = a_k x_k$, $k = 1, 2, 3$, it follows from (3.3) that

$$\psi^\circ(P;\ F) = \frac{1}{(2\pi)^{3/2}} \frac{1}{a_1a_2a_3} \int\limits_{S^*} \exp\{i\,(P_1T)\}\, dv_T. \tag{4.2}$$

Here S^* is the sphere of unit radius

$$t_1^2 + t_2^2 + t_3^2 \leqslant 1,\ P_1 = (x_1 / a_1,\ x_2 / a_2,\ x_3 / a_3),\ T = (t_1,\ t_2,\ t_3),\ dv_T = dt_1,\ dt_2\ dt_3.$$

Let us convert from Cartesian t_1, t_2, t_3 to spherical r, φ, ϑ coordinates; here ϑ is the angle between OP_1 and OT. As a result we obtain

$$\psi^\circ(P;\ F) = \frac{1}{a_1a_2a_3} \frac{1}{R^{3/2}} J_{3/2}(R), \qquad R^2 = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2}. \tag{4.3}$$

We seek the charged particle distribution in the ionized gas cloud in the form

$$d\mu^* = \chi^*(\ |\ a_1^2y_1^2 + a_2^2y_2^2 + a_3^2y_3^2\ |\ )\ \ (Q = (y_1,\ y_2,\ y_3),\ Q \in F). \tag{4.4}$$

For the Fourier transform $\chi^\circ(P)$ of the charged-particle distribution density $\chi(Q)$ that

$$\chi^\circ(P) = \frac{1}{(2\pi)^{3/2}} \int\limits_D \chi\,(|\,a_1^2y_1^2 + a_2^2y_2^2 + a_3^2y_3^2\,|)\ e^{i\,(PQ)}\, dv_Q\,,$$
$$D\colon \{a_1^2x_1^2 + a_2^2x_2^2 + a_3^2x_3^2 \leqslant 1\}. \tag{4.5}$$

Substituting $t_k = a_k y_k$, $\zeta^2 = a_1^2y_1^2 + a_2^2y_2^2 + a_3^2y_3^2$ and converting to spherical coordinates, we find

$$\chi^\circ(P) = (a_1a_2a_3)^{-1}\chi^\circ(R). \tag{4.6}$$

Here $\chi^\circ(R)$ is the Hankel transform of the unknown distribution density $\chi(\zeta)$,

$$\chi^\circ(R) = \frac{1}{\sqrt{R}} \int\limits_0^1 \zeta^{3/2}\chi(\zeta)\, J_{1/2}(\zeta R)\, d\zeta. \tag{4.7}$$

As a result the operator equation (3.5) for the case in question is recast in the form

$$W(F)\, R^{-3/2} J_{3/2}(R)\, K^{-1/2} = (2\pi)^{3/2} \chi^\circ(R)\, \varphi^\circ(\rho). \tag{4.8}$$

For further analysis it is convenient to transform the operator equation (4.8). Let

$$x_k = Ra_kt_k \qquad (k = 1,2,3),$$
$$R^2 = x_1^2 / a_1^2 + x_2^2 / a_2^2 + x_3^2 / a_3^2,\ \ t_1^2 + t_2^2 + t_3^2 \leqslant 1.$$

Since $\rho^2 = R^2(a_1^2t_1^2 + a_2^2t_2^2 + a_3^2t_3^2)$, then

$$W(F)R^{-3/2}J_{3/2}(R)K^{-1/2} = (2\pi)^{3/2}\chi^\circ(R)\varphi^\circ(R[a_1^2t_1^2 + a_2^2t_2^2 + a_3^2t_3^2]^{1/2}). \tag{4.9}$$

After integrating both sides of (4.9) over the surface of the sphere of unit radius $t_1^2 + t_2^2 + t_3^2 = 1$ we find

$$K^{-1/2}SW(F)R^{-3/2}J_{3/2}(R) = (2\pi)^{3/2}\chi^\circ(R)\varphi^{\circ *}(R). \tag{4.10}$$

Here S is the surface area of the unit sphere S^* and

$$\varphi^{\circ *}(R) = \int_{S^*} \varphi^{\circ}(R[a_1^2 t_1^2 + a_2^2 t_2^2 + a_3^2 t_3^2]^{1/2})\, d\sigma \qquad (4.11)$$

($d\sigma$ is an element of the surface S^*.)

For the existence of a Lyapunov-stable equilibrium charged-particle distribution in the ionized gas cloud it is sufficient that there exist the function $\chi(\zeta)\zeta^{1/2}$ whose Hankel transform is the function $\chi^{\circ}(R)R^{1/2}$, which is the solution of the operator equation (4.10).

5. For concreteness let us find the Hankel original $\chi^*(\zeta)$ for the selected potential function $\varphi(r)$. Substituting (1.2) into (3.4), we find

$$\varphi^{\circ}(\rho) = \sqrt{2/\pi}\, \lambda^2/\rho^2(\lambda^2 + \rho^2). \qquad (5.1)$$

Now we obtain for $\varphi^{\circ *}(R)$

$$\varphi^{\circ *}(R) = 4\pi R^{-2} \sqrt{2/\pi}\, B(R). \qquad (5.2)$$

Here, depending on the relationship between the axes of the ellipsoid of revolution F, we have

$$\text{a) } B(R) = \frac{1}{2a\beta} \ln \frac{1+\beta/a}{1-\beta/a} - \frac{R}{2\alpha(R)\beta} \ln \frac{1+\beta R/\alpha(R)}{1-\beta R/\alpha(R)} \quad \text{for } a > c, \qquad (5.3)$$

$$\text{б) } B(R) = \frac{1}{a\beta} \operatorname{arc\,tg} \frac{\beta}{a} - \frac{R}{\alpha(R)\beta} \operatorname{arc\,tg} \frac{\beta R}{\alpha(R)} \quad \text{for } a < c, \qquad (5.4)$$

$$\alpha(R) = (\lambda^2 + a^2R^2)^{1/2}.$$

(If $a > c$, then $\beta^2 = a^2 - c^2$; if $a < c$, then $\beta^2 = c^2 - a^2$. We note that for $a > c$ an inequality of the form $(\beta R/\alpha(R) < 1)$ is satisfied for all R from 0 to $+\infty$.)

With the aid of the operator equation (4.10), after inverse Hankel transformation of order 1/2 we find the expression for the Lyapunov-stable equilibrium distribution density:

$$K^{1/2}\chi(\zeta) = \frac{W(F)}{4\pi\zeta^{1/2}} \int_0^{\infty} \frac{R^2}{B(R)} J_{3/2}(R)\, J_{1/2}(\zeta R)\, dR \qquad (5.5)$$

The normalization condition for $\chi(\zeta)$ yields the total number $\mu(F)$ of charged particles enclosed in the volume of the ionized-gas cloud:

$$\mu(F) = W(F)\omega(F). \qquad (5.6)$$

Here $1/\omega(F)$ is the chemical potential for the considered case of an ionized gas forming a cloud of elliptic form (the introduction of the upper limit R_{max} corresponds to the concept of finite radius of the ionized gas particles):

$$\omega(F) = \frac{1}{a^2 c K^{1/2}} \int_0^{R_{max}} R\,[J_{3/2}(R)]^2 \frac{dR}{B(R)}. \qquad (5.7)$$

It follows from the normalization condition for $\chi(\zeta)$, in particular, that $K^{1/2}$ equals the total number of charged particles in the ionized-gas cloud, i.e., $K^{1/2} = \mu(F)$.

6. Now let n be the average neutral molecule concentration, then for an ionization degree f the charged particle concentration in the gas constitutes fn and the total number $\mu(F)$ of charged particles in the ionized gas cloud having the volume $4/3\pi a^2 c$ will be $\mu(F) = 4/3\pi a^2 c f n$. Equating the two values for $\mu(F)$, we find the magnitude of the potential energy

$$W(F) = \frac{4\pi}{3} \frac{a^2 c f n}{\omega(F)}. \qquad (6.1)$$

We divide the energy W(F) by the volume of the ionized-gas cloud and obtain the gas pressure $p(F) = fn/\omega(F)$ inside the cloud.

Equilibrium is established when the ionized gas pressure becomes equal to the external pressure, specifically when $p(F) = p_0$, where p_0 is the external pressure. Hence we find the expression for the degree of ionization $f(F)$

$$f(F)=\frac{1}{2a^2cn}\left(\frac{3p_0}{\pi}\int\limits_0^{R_{max}} R\,[J_{3/2}(R)]^2\frac{dR}{B(R)}\right)^{1/2}, \tag{6.2}$$

which depends on the ionized gas cloud form F, its dimensions a and c, the magnitude of the external pressure p_0, and temperature of the ionized gas, which enter through the parameter λ. In particular, it can be shown that for a highly rarefied gas the parameter λ^{-1} is equal to the short-range interaction parameter introduced by Landau in [11], i.e., $\lambda^{-1} = e^2/\theta$, where θ is the average temperature of the ionized gas; e is the elementary charge.

7. We use a series expansion to obtain the approximate expressions for the Lyapunov-stable equilibrium charged-particle distribution density $\chi(\zeta)$ in an ionized-gas cloud.

Let $a > c$. We expand $1/B(R)$ into a series using the geometric progression formula with common ratio equal to

$$\frac{R}{2b^2\beta\alpha(R)}\ln\frac{1+\beta R/\alpha(R)}{1-\beta R/\alpha(R)}\qquad\left(b^2=\frac{1}{2a\beta}\ln\frac{1+\beta/a}{1-\beta/a}\right).$$

Then

$$\frac{1}{B(R)}=\sum_{k=0}^{\infty}\frac{1}{2^k b^{2k+2}\beta^k}\left(\frac{R}{\alpha(R)}\right)^k\left[\ln\frac{1+\beta R/\alpha(R)}{1-\beta R/\alpha(R)}\right]^k. \tag{7.1}$$

After further expansion into a power-law series of the logarithm appearing in (7.1), retaining only the first terms of the expansion, we find

$$\frac{1}{B(R)}\approx\frac{1}{b^2}-\frac{R^2}{b^4[\alpha(R)]^2}, \tag{7.2}$$

We substitute into (5.5) the approximate expression for $1/B(R)$, for the condition $a \gg c$, since

$$J_{1/2}(\zeta R)=\sqrt{2/\pi R\zeta}\sin R\zeta,\quad J_{3/2}(R)=\sqrt{2/\pi R}\,(R^{-1}\sin R-\cos R);$$

we obtain the equilibrium charged-particle distribution density in the "constricted" discharge of finite dimensions:

$$\chi(\zeta)\approx\frac{W(F)}{4\pi}\left(\frac{1}{2\ln(2a/c)}\right)\frac{\delta(1-\zeta)}{\zeta}+\\+\frac{W(F)}{4\pi}\left(1+\frac{a}{\lambda}\right)\left(\frac{\lambda}{a}\right)^2\left(\frac{1}{2\ln(2a/c)}\right)e^{-\lambda/a}\left(\frac{\mathrm{sh}(\lambda\zeta/a)}{\zeta}\right). \tag{7.3}$$

Thus the equilibrium charged-particle distribution density in the "pinch" increases toward the periphery, which agrees with the conclusion of Vlasov [12] obtained as a result of hydrodynamic analysis. It was shown in [12] in particular that the ionized gas cloud of elliptic form has hydrodynamic stability with respect to small perturbations of form.

Now let $a < c$. We use the expression for B(R) which is valid for $a < c$. We divide in (5.5) the entire interval of integration with respect to R from 0 to $+\infty$ into two intervals: from 0 to R_1 and from R_1 to $+\infty$. We select R_1 so that the inequality $(\beta R/\alpha(R) < 1$ is satisfied for $0 \le R < R_1$. This is possible for $R_1 = \lambda/(\beta^2 - a^2)^{1/2}$.

Now, after corresponding expansions of B(R) into series, from (5.5) follows the approximate expression for $\chi(\zeta)$ for $a < c$. This expression includes a linear combination of the products of trigonometric functions by the corresponding integral sine and integral cosine. For $\zeta = 0$ the equilibrium charged-particle distribution density $\chi(\zeta)$ has a maximum and decreases as $\zeta \to 1$. The resultant approximate expression for $\chi(\zeta)$ for $a < c$ is not presented here because of its complexity.

We note that expression (5.5) for $\chi(\zeta)$ is exact and admits integration by numerical methods, for example, the Simpson method [13, 14].

We have examined the case of "smoothed" Coulomb interaction of charged particles in an ionized gas without account for uniform rotation of the entire cloud about one of the ellipsoid axes. However, there are no fundamental difficulties in a more detailed account for short-range interaction (for example, with the aid of the Lennard-Jones potentials (see [4])), nor in accounting for the effect of uniform rotation of the ionized gas cloud as a whole.

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